The Role of Nonlinear Processes in the Generation of Gravity Waves by Convective Clouds

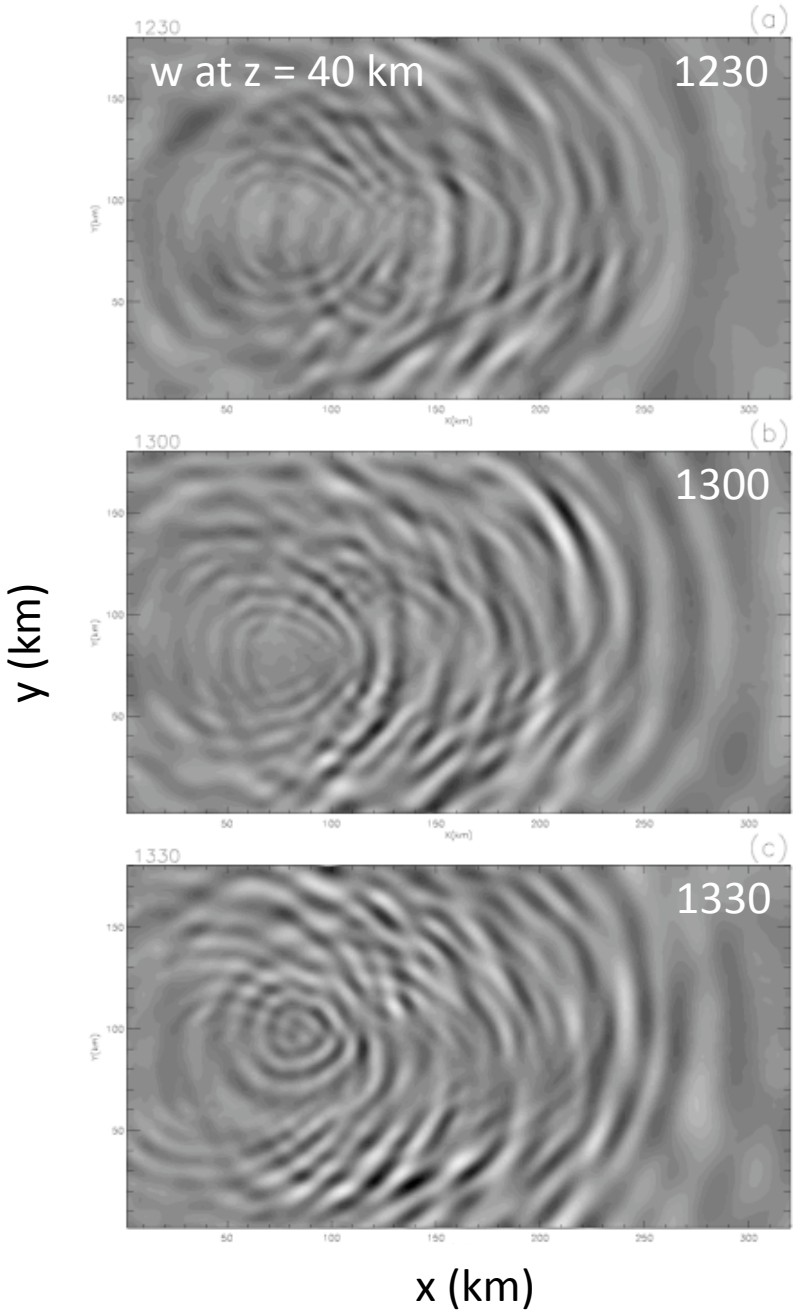
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Background

- A number of previous studies (by some of the people this room) have sought to understand the generation mechanisms for convectively generated gravity waves in cloud models.

- The vertical velocity at a height of 40 km is shown at (a) 1230 LST, (b) 1300 LST and (c) 1330 LST. (From Lane, Reeder and Clarke., 2001, JAS, 58, 1249 - 1274.)



Background

- Past numerical studies of convectively-generated waves examined the relative importance of diabatic heating and the internal circulation of the cloud in the generation of gravity waves.

- These studies have been used to evaluate the hypothesized physical mechanisms: diabatic heating, the obstacle effect and the mechanical oscillator.

- While such models have a (reasonably) complete representation of the physics, the numerical solution can be very difficult to interpret.

Aim

- The present study revisits the problem of how gravity waves are generated by convection.

- The aim is to understand the relative importance of diabatic heating and the internal convective circulation in a very (simple-minded) idealized, (almost) analytically tractable model.

Governing Equations

- Write the equations with the linear terms on the lhs and the nonlinear terms on the rhs.

$$\frac{\partial u}{\partial t} + \frac{\partial \phi}{\partial x} = F_u \qquad \text{where} \qquad F_u = -\left(\frac{\partial u^2}{\partial x} + \frac{\partial (uw)}{\partial z} \right)$$

$$\frac{\partial w}{\partial t} + \frac{\partial \phi}{\partial z} - b = F_w \qquad F_w = -\left(\frac{\partial (uw)}{\partial x} + \frac{\partial w^2}{\partial z} \right)$$

$$\frac{\partial b}{\partial t} + N_a^2 w = \dot{Q} + F_b \qquad F_b = -\left(\frac{\partial (ub)}{\partial x} + \frac{\partial (wb)}{\partial z} \right)$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0$$

Nonlinear Wave Equation

- Form a single equation for the streamfunction ψ

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + N_a^2\frac{\partial^2\psi}{\partial x^2} = -\frac{\partial \dot{Q}}{\partial x} + G$$

where $\quad u = \frac{\partial \psi}{\partial z} \qquad w = -\frac{\partial \psi}{\partial x} \qquad \eta = \nabla^2\psi$

$$G = \frac{\partial}{\partial t}\left[\frac{\partial F_u}{\partial z} - \frac{\partial F_w}{\partial x}\right] - \frac{\partial F_b}{\partial x} = -\nabla\cdot\left\{\frac{\partial}{\partial t}\left(\mathbf{u}\nabla^2\psi\right) + \frac{\partial}{\partial x}\left(\mathbf{u}b\right)\right\}$$

Linearized Wave Equation

- Nonlinear wave equation

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + N_a^2\frac{\partial^2\psi}{\partial x^2} = -\frac{\partial \dot{Q}}{\partial x} + G$$

- Linearize by setting $G = 0$

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi_L + N_a^2\frac{\partial^2\psi_L}{\partial x^2} = -\frac{\partial \dot{Q}}{\partial x}$$

where $\quad u_L = \frac{\partial \psi_L}{\partial z} \qquad w_L = -\frac{\partial \psi_L}{\partial x}$

Periodic Heating

- Unbounded domain
- Assume the heating takes the form

$$\dot{Q}(x,z,t) = \begin{cases} Q_0 \exp\left(-\frac{x^2}{\sigma^2}\right)\cos\left(\frac{\pi z}{h}\right)\cos(\Omega t) & |z| \le \frac{h}{2} \\ 0 & \text{otherwise} \end{cases}$$

- Q_0 = 25 K/day, σ = 10 km, h = 10 km, Ω = 2π/(2 hours)

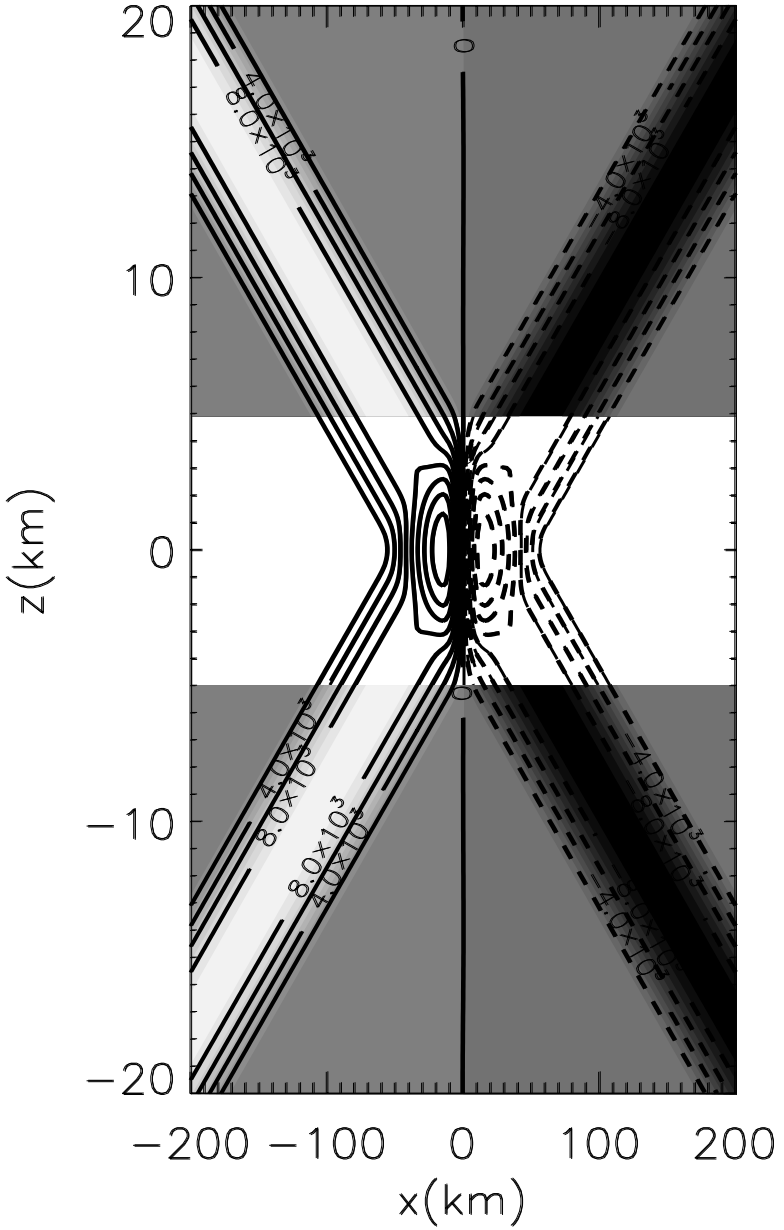
Linear Solution

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi_L + N_a^2\frac{\partial^2\psi_L}{\partial x^2} = -\frac{\partial \dot{Q}}{\partial x}$$

- Assume solutions of the form

$$\psi(x,z,t) = \hat{\psi}(x,z)\cos(\Omega t)$$

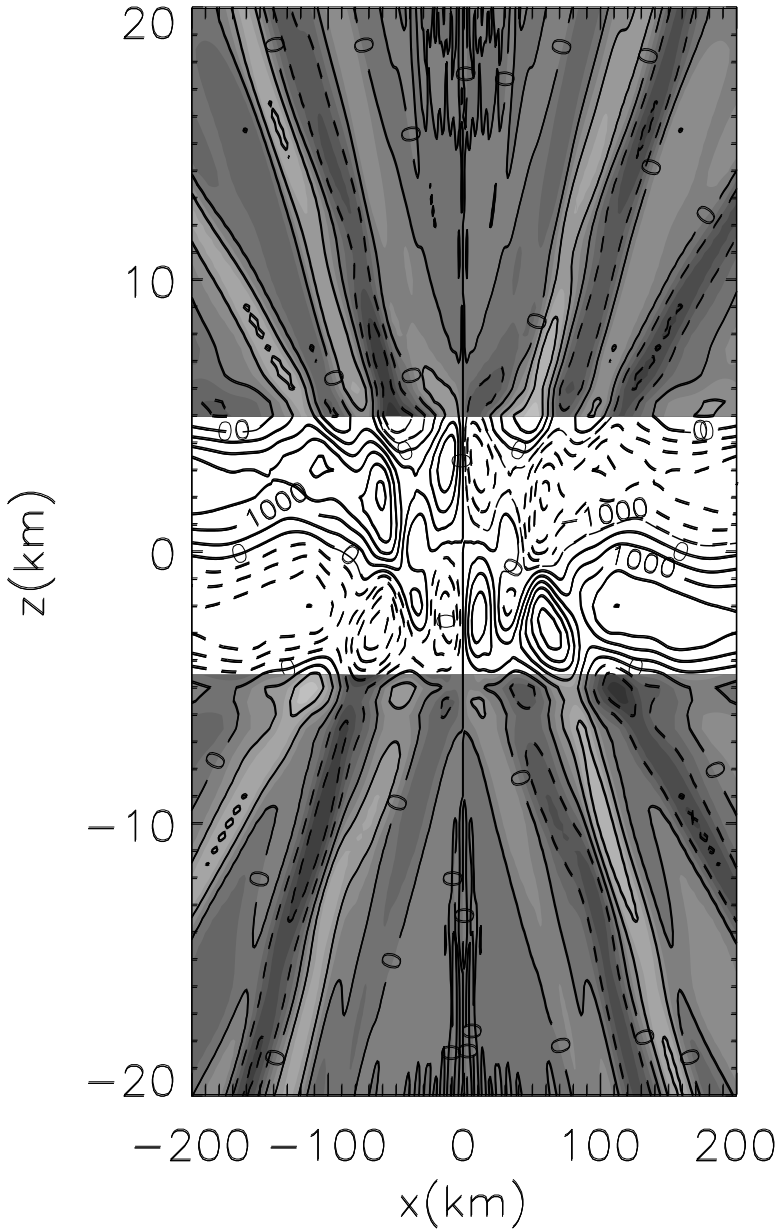
- Analytic solution for the streamfunction at 2 hrs (1 period).



NL Solution

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi + N_a^2\frac{\partial^2\psi}{\partial x^2} = -\frac{\partial \dot{Q}}{\partial x} + G$$

- Difference between the nonlinear solution and linear solution for the streamfunction at 2 hrs (1 period).
- Forcing confined between –h/2 and h/2.



Weakly Nonlinear Wave Equation

- Subtract the linear wave equation from the nonlinear wave equation.

- Approximate $G \approx G_L$

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi_{WNL} + N_a^2\frac{\partial^2\psi_{WNL}}{\partial x^2} = G \approx G_L$$

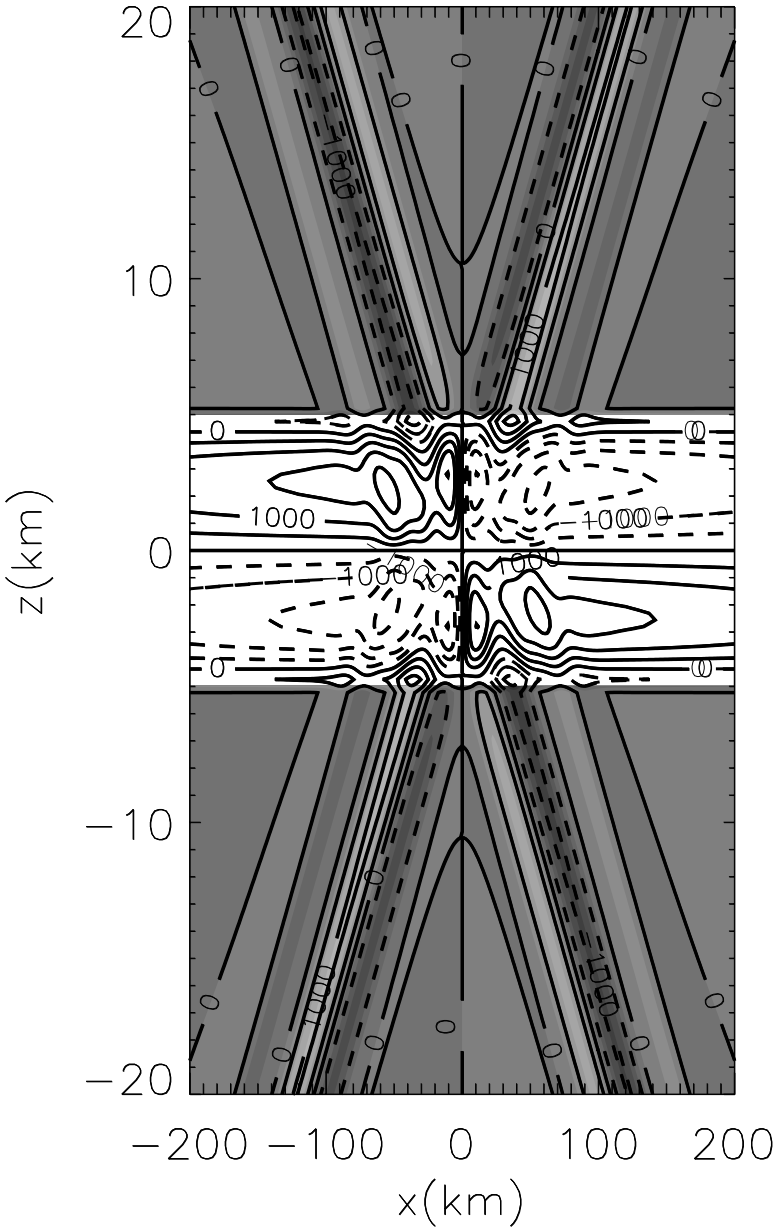
where $$u_{WNL} = \frac{\partial\psi_{WNL}}{\partial z} \qquad w_{WNL} = -\frac{\partial\psi_{WNL}}{\partial x}$$

and $$G_L = -\nabla\cdot\left\{\frac{\partial}{\partial t}\left(\mathbf{u}_L\nabla^2\psi_L\right) + \frac{\partial}{\partial x}\left(\mathbf{u}_L b_L\right)\right\}$$

WNL Solution

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi_{WNL} + N_a^2\frac{\partial^2\psi_{WNL}}{\partial x^2} = G_L$$

- Analytic WNL solution for the streamfunction at 2 hrs (1 period).
- Forcing confined between –h/2 and h/2.
- Higher frequency waves.
- Steady solution.



Forcing

- Consider G_L

$$G_L = -\nabla \cdot \left(\frac{\partial}{\partial t}\left(\mathbf{u}_L \nabla^2 \psi_L \right) - \frac{\partial}{\partial x}\left(\mathbf{u}_L b_L \right) \right)$$

- The rhs involves products of terms of the form

$$\psi_L\left(x,z,t\right) = \hat{\psi}\left(x,z\right)\cos\left(\Omega t\right)$$

- This leads to terms involving

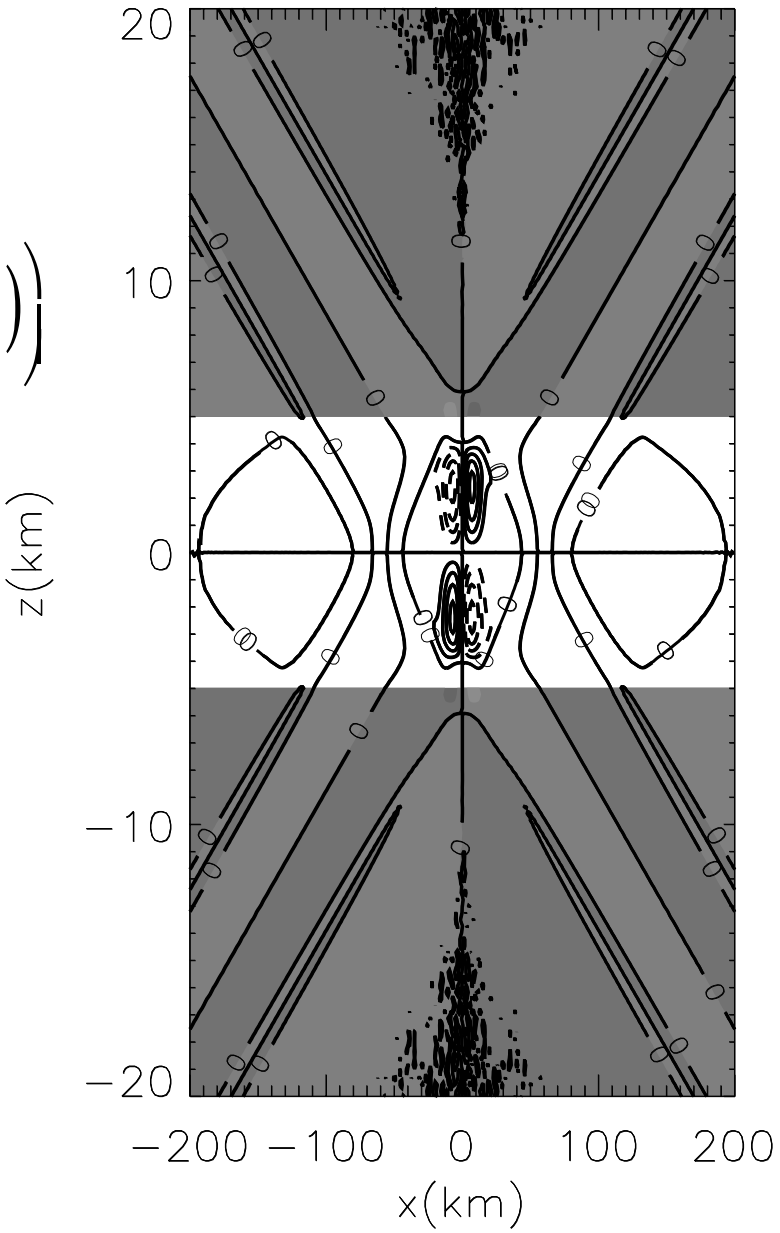
$$\cos^2\left(\Omega t\right) = \frac{1}{2}\left(1+\cos\left(2\Omega t\right)\right)$$

- G forces waves at $\omega = 2\Omega$ and a steady overturning circulation $\omega = 0$.

Forcing Function

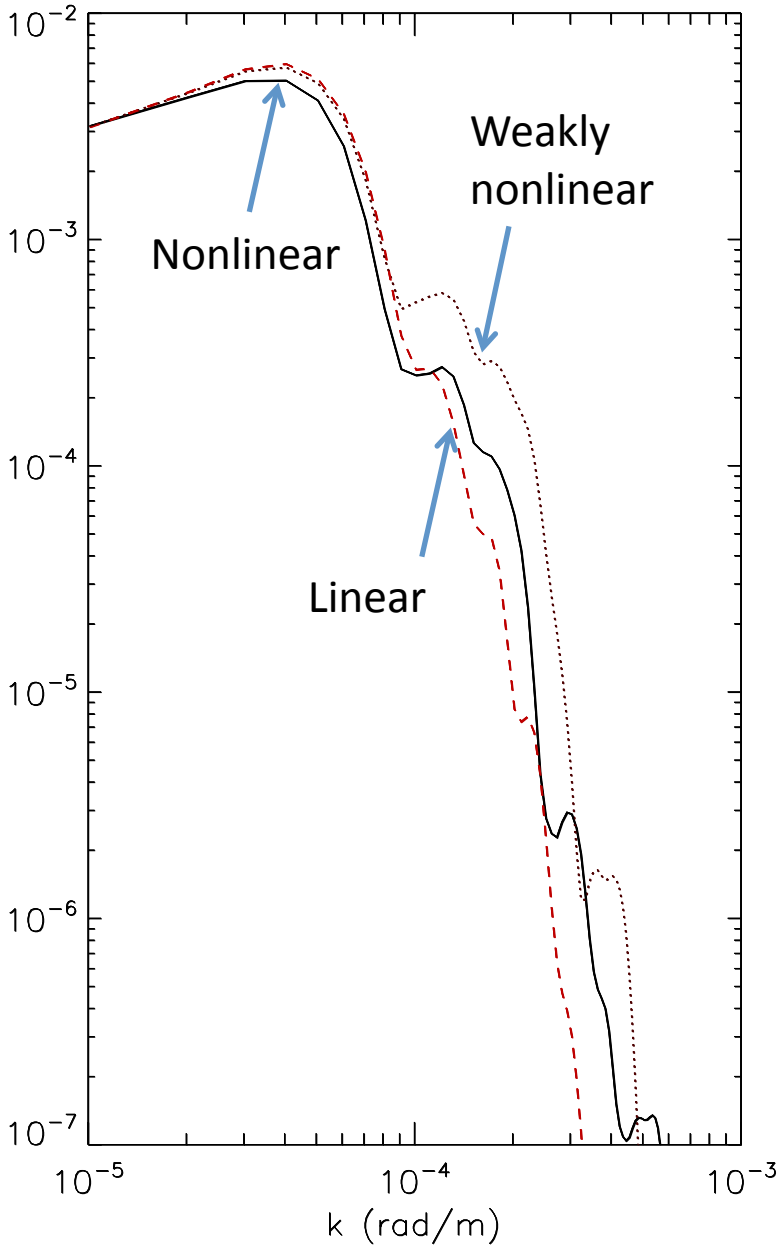
$$G_L = -\nabla \cdot \left(\frac{\partial}{\partial t}\left(\mathbf{u}_L \nabla^2 \psi_L \right) + \frac{\partial}{\partial x}\left(\mathbf{u}_L b_L \right) \right)$$

- G_L is a compact, quadrapole.
- Reminiscent of Lighthill's acoustic radiation problem (but not spontaneous emission).



Power Spectrum

- Spectrum of momentum flux at 2 hrs (1 period).
- Lower power at lower wavenumber in the nonlinear calculation.
- Increased power at higher wavenumber in the nonlinear and weakly nonlinear calculations.
- Spectrum broadened too much in the weakly nonlinear calculation.



Periodic Heating

- Half plane $z \geq 0$.
- Assume solutions of the form

$$\psi(x,z,t) = \hat{\psi}(x,z)\cos(\Omega t)$$

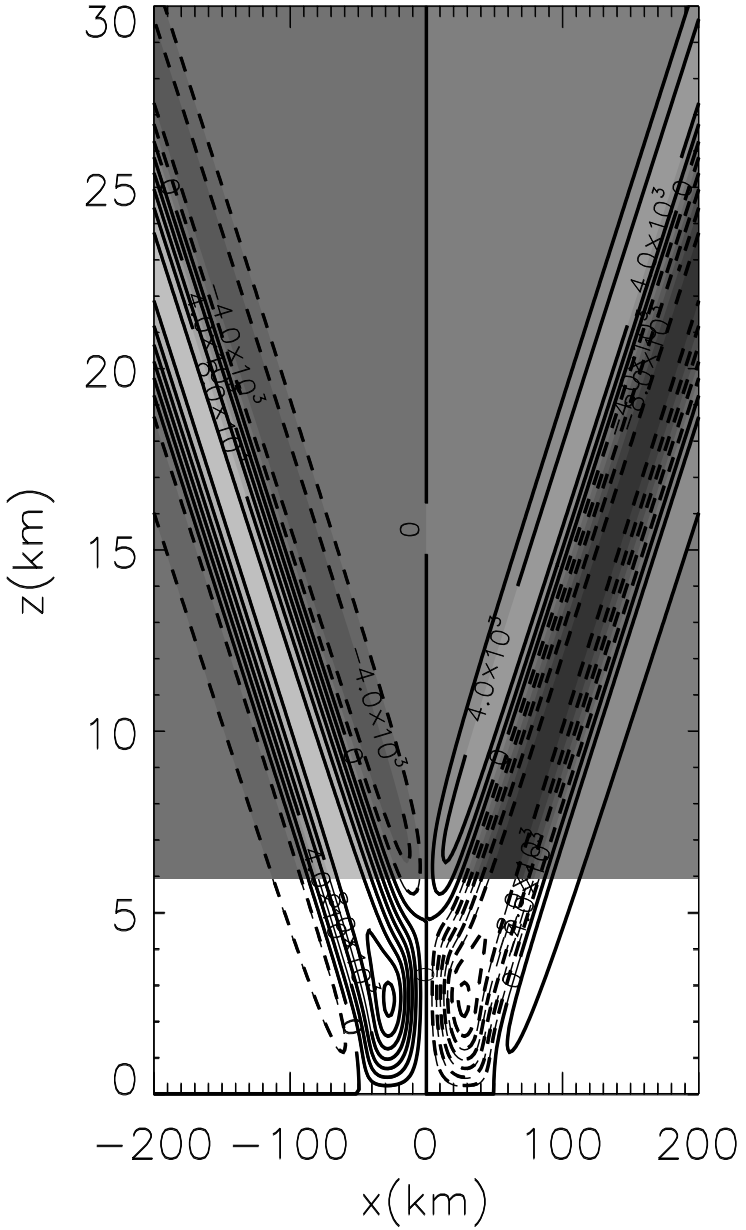
$$Q(x,z,t) = \begin{cases} Q_0 \exp\left(-\frac{x^2}{\sigma^2}\right)\sin\left(\frac{\pi z}{h}\right)\cos(\Omega t) \\ 0 \end{cases}$$

- Q_0 = 25 K/day

Linear Solution

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi_L + N_a^2\frac{\partial^2\psi_L}{\partial x^2} = -\frac{\partial \dot{Q}}{\partial x}$$

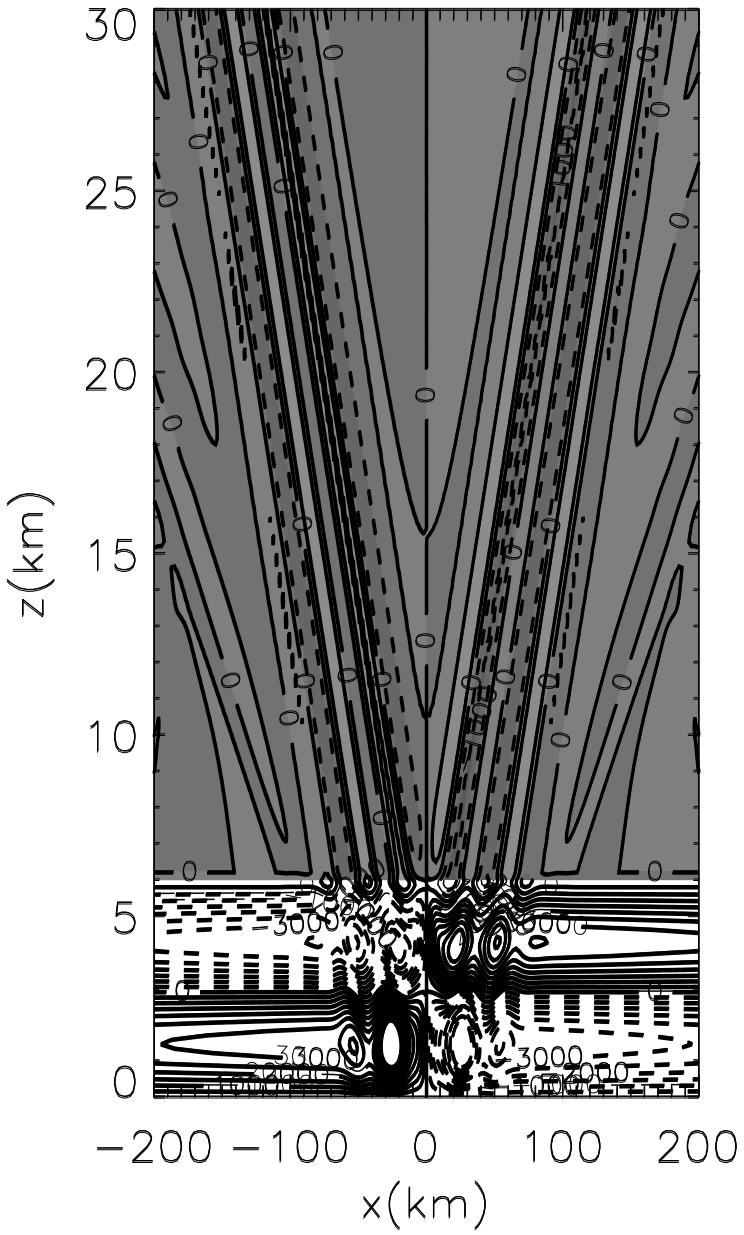
- Analytic solution for the streamfunction at 0.45 hr.



WNL Solution

$$\frac{\partial^2}{\partial t^2}\nabla^2\psi_{WNL} + N_a^2\frac{\partial^2\psi_{WNL}}{\partial x^2} = G_L$$

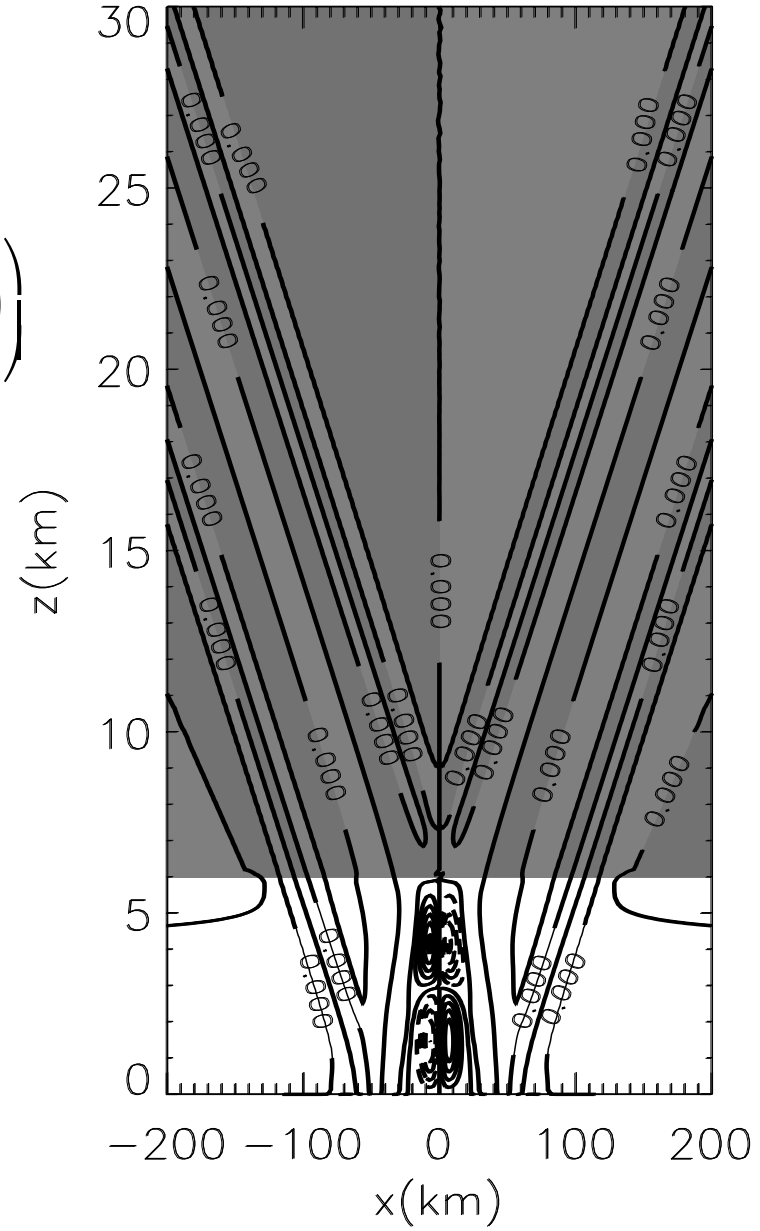
- Analytic WNL solution for the streamfunction at 0.45 hr.
- Higher frequency waves.
- Steady solution.



Forcing Function

$$G_L = -\nabla \cdot \left\{ \frac{\partial}{\partial t}\left(\mathbf{u}_L \nabla^2 \psi_L \right) + \frac{\partial}{\partial x}\left(\mathbf{u}_L b_L \right) \right\}$$

- G_L is a compact, quadrapole.
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Summary and Conclusions

- The linear solution is the response to imposed periodic heating rate Q with frequency $\omega = \Omega$.
- The nonlinear correction is the response to the internal circulation of the cloud represented by G.
- The weakly nonlinear correction, constructed by assuming $G \approx G_L$, is a reasonable, qualitative approximation to the fully nonlinear correction.
- G_L has a quadrapole structure.

Conclusions

- G forces waves at $\omega = 2\Omega$ and a steady overturning circulation $\omega = 0$.
- G forces waves at relatively higher wavenumber k.
- The nonlinear forcing increases the integrated momentum flux.
- Like a number of previous studies the linear calculations overestimate the amplitude of waves when compared with the equivalent nonlinear calculations.

Conclusions

- The weakly nonlinear calculation broadens the spectrum, but overestimates the power at all wavenumbers.

- Important caveat - the calculations have neglected transients and shear.